University of North Carolina at Charlotte
Mathematical Finance Program
Comprehensive Exam

Fall, 2008

Directions: This exam consists of 7 questions. Please answer each question. This exam begins at 1:00 pm on Friday November 21st, and ends exactly at 4:00 pm.

No exam will be accepted after 4:00 pm. You will be given a 5 minute warning at 3:55, and a 1 minute "last call" at 3:59. The proctor is not allowed to accept exams after 4:00 pm exactly.

Do not write your name on your answer sheet or on any exam page other than this cover sheet. You have been assigned an identification letter below. Write this identification letter on each of your answer pages *and* on each of the test sheets.

Please note that we will not accept any answer sheet pages with your name on them; if your name is on the sheet we will not grade it.

Your identification letter for this exam is: ________________________

Academic Honesty Statement: All students must comply with University policies on academic integrity. Any student violating these policies, as defined on pages 262-263 of the Graduate Catalog, will be referred to the University administration for disciplinary action. Sanctions for academic dishonesty include, but are not limited to, failure of this exam, suspension, or expulsion from the University. By signing below you certify that the answers you place on this exam are your own and that you have not received help from others.

I hereby certify that the work on this exam is my own. I also certify that I am aware that the test cannot be turned in after 4:00 pm and that I must place my exam identification letter on my exam sheets.

Student name (print) __________________________________

Student Signature: ____________________________________

Exam Identification Letter: _____________

1.

Using annual data from 1960 to 1991 you estimate the relationship between real wages per hour (Y) and output per hour (X) for the U.S. business sector (standard errors in parentheses):

(eq 1) Y_t	$=$	18.09	$+$	0.815 X_t			$R^2 = 0.946$
		(3.31)		(0.035)			Durbin-Watson d $= 0.448$
(eq 2) Y_t	$=$	1.33	$+$	0.143 X_t	$+$	0.895 Y_{t-1}	$R^2 = 0.985$
		(2.31)		(0.063)		(0.177)	Durbin's h=1.943

(a) Does autocorrelation appear to be a problem in equation (1)? Explain.

(b) Derive an estimate of the autocorrelation coefficient (ρ) from equation (1).

(c) What is the motivation for estimating equation (2)? What is the possible problem with including the lagged dependent variable?

(d) What if you found that both Y and X are non-stationary. Are there any implications for the results in equation (1)?

2.

The GARCH(1,1) model for financial returns r_t can be represented as follows:

$$r_t = \sqrt{h_t}\eta_t$$

$$\eta_t \sim NIID(0,1)$$

$$h_t = a_0 + a_1 r_{t-1}^2 + b_1 h_{t-1}$$

$$a_0 > 0, a_1 > 0, b_1 > 0$$

(a) Assuming constant unconditional variance, $E(r_t^2) = \sigma^2$, derive σ^2 in terms of the parameters of the model a_0, a_1 and b_1. Under what assumptions on the parameters is the variance σ^2 finite?

(b) Intuitively, how can the above GARCH model explain volatility clustering in returns?

(c) How can the above GARCH model generate fat-tails often displayed by financial returns? [hint: derive the kurtosis for the form $\sqrt{h_t}\eta_t$, assume $Var(h_t) > 0$ while ignoring the dynamics in the h_t process]

3. Answer the following questions:

a) Explain why you agree or disagree with the following statement: "It is always better to have a portfolio with more convexity than one with less convexity."

b) Explain why you agree or disagree with the following statement: "A bullet portfolio will always outperform a barbell portfolio with the same dollar duration if the yield curve steepens."

c) What is a laddered portfolio?

d) You are given the following information on three bonds:

Maturity	YTM	Bond Price	$ Duration	Quantity
2 years	5.5%	100.6882	195.01	q_S
5 years	6.5%	98.3655	450.14	-1,000
10 years	8%	97.9847	782.18	q_L

We want to construct a butterfly by selling 1,000 contracts of 5-year bonds and buying q_s of 2-year contracts and q_L of 10-year bonds (q_s and q_L represent number of contracts). (24 points)

i) The first approach is to create a cash and dollar duration neutral butterfly (i.e., total cash outflow in the long position equals the total cash inflow in the short position AND dollar duration on the long equals the dollar duration on the short.) What are q_s and q_L?

Use the maturity weighing approach to build your butterfly. In other words, the beta would be (maturity on the middle bond – maturity on the short bond)/(maturity on the long bond – maturity on the middle bond). What would be the q_s and q_L?

4.

Consider a 1-period economy with 2 states and 2 securities. The first security is a bond that pays out 10 in each state at period's end and the second security pays out 20 in the first state and 10 in the second state at period's end. The price of the bond is 8 and the price of the second security is 10.

a. Is the market complete? Explain.
b. Do equivalent Martingale measures exist? Find them all.

Find the price of a European call option on Security 2 with strike price 15. How many units of each security should be bought or sold to replicate the payout of this call option?

5.

Suppose your friend thinks that gold, which is currently \$900 per ounce, will go up in price to \$1000 per ounce over the next year. He is so sure of his forecast that he is willing to buy gold from you one year from today for \$990 per ounce. The constant risk-free rate is 5% and the storage cost per ounce of gold is \$30 per year payable semi-annually in advance.

Show that your friend is presenting you with an arbitrage opportunity. In particular, by trading only gold and borrowing/lending at the risk-free rate, you can make a riskless profit at your friend's expense. Carefully describe the trading strategy and find the riskless profit.

6.

The American call option is the solution of the following linear complementarity problem on a finite domain:

$$\begin{cases} \min\left(\dfrac{\partial \overline{V}}{\partial \tau} - \mathbf{L}_\xi \overline{V}, \overline{V}(\xi,\tau) - \max(2\xi - 1, 0)\right) = 0, \quad 0 \le \xi \le 1, \quad 0 \le \tau, \\ \overline{V}(\xi, 0) = \max(2\xi - 1, 0), \quad 0 \le \xi \le 1, \end{cases}$$

where

$$\mathbf{L}_\xi = \frac{1}{2}\sigma^2(\xi)\xi^2(1-\xi)^2\frac{\partial^2}{\partial \xi^2} + (r - D_0)\xi(1-\xi)\frac{\partial}{\partial \xi} - [r(1-\xi) + D_0\xi].$$

Reformulate this problem as a free-boundary problem if $D_0 > 0$.

7. Answer **one** of the following two questions. (If you answer both, only the first will be graded.)

Problem 1. Let $(W_t)_{t\ge 0}$ be a Brownian motion defined on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$. Define $X_t = W_t^4 - 6tW_t^2 + 3t^2$.

1. Show that X_t is a martingale.
2. Show that X_t is a Markov process.
3. Compute the quadratic variation $[X, X]_t$.

Problem 2. Let S_n be the stock price process in a Binomial tree, i.e., $S_{n+1} = uS_n$ if ω_n =Head and $S_{n+1} = dS_n$ if ω_n =Tail where u and d are constants.

1. Write down the no-arbitrage condition for this Binomial model assuming interest rate is a constant r.
2. Write down the recursive risk neutral pricing formula for an option with payoff $f(S_N)$ at time N.
3. Write down the σ-algebra generated by S_2 and compute the conditional expectation $E[S_3|S_2]$.