University of North Carolina at Charlotte
Mathematical Finance Program
Comprehensive Exam

Spring, 2008

Directions: This exam consists of 6 questions. Please answer each question. This exam begins at 10:00 pm on Friday April 4th, and ends exactly at 1:00 pm.

<u>No exam will be accepted after 1:00 pm. You will be given a 5 minute warning at 12:55, and a 1 minute "last call" at 12:59. The proctor is not allowed to accept exams after 1:00 pm exactly.</u>

Do not write your name on your answer sheet or on any exam page other than this cover sheet. You have been assigned an identification letter below. Write this identification letter on each of your answer pages *and* on each of the test sheets.

<u>Please note that we will not accept any answer sheet pages with your name on them; if your name is on the sheet we will not grade it.</u>

Your identification letter for this exam is: ______________________

Academic Honesty Statement: All students must comply with University policies on academic integrity. Any student violating these policies, as defined on pages 262-263 of the Graduate Catalog, will be referred to the University administration for disciplinary action. Sanctions for academic dishonesty include, but are not limited to, failure of this exam, suspension, or expulsion from the University. By signing below you certify that the answers you place on this exam are your own and that you have not received help from others.

I hereby certify that the work on this exam is my own. I also certify that I am aware that the test cannot be turned in after 1:00 pm and that I must place my exam identification letter on my exam sheets.

Student name (print) ________________________________

Student Signature: __________________________________

Exam Identification Letter: _____________

1. Consider the linear regression model y = Xβ + ε, which satisfies the full ideal conditions.
   a. List the full ideal conditions (FIC).
   b. Given the FIC, $\hat{\beta}$ and $\hat{\sigma}^2$ have certain desirable properties. List and briefly describe each.
   c. What is the OLS estimator of $\hat{\beta}$ and what mathematical problem does $\hat{\beta}$ solve?
   d. Which desirable properties are lost with the following failures in the full ideal conditions?
      i. Right hand side variables are stochastic
      ii. There are different variances across ε_i .
      iii. There is dependence of ε_t on ε_{t-1}

2. Let C_1 be the price of an American call option with strike price K_1 and let C_2 be the price of an otherwise identical American call option but with strike price K_2 .
   a. Show $C_1 - C_2 \leq K_2 - K_1$
   b. Show that if it is optimal to exercise the K_2 call at time *t*, then it is also optimal to exercise the K_1 call at time *t*.

3. You observe the following default-free noncallable term structure of interest rates:

	Maturity (yrs)	Coupon	Yield (APR)	Price	Modified duration	Macaulay duration	Convexity
T bill	1	0.000%	5.81%	$94.434	0.97	1.00	1.417
T note	2	6.250%	6.18%	$100.130	1.85	1.91	4.426
T note	3	6.375%	6.34%	$100.094	2.69	2.78	8.895
T note	5	6.625%	6.56%	$100.273	4.20	4.34	21.278
T note	10	7.000%	6.79%	$101.507	7.14	7.38	64.694
T bond	30	6.000%	6.97%	$87.865	12.84	13.29	264.667

   a. You are a manager of a $100 million pension fund and can only invest in the bonds listed above. You want a portfolio with Macaulay duration of 5 years. There are many ways using these securities to create portfolios with Macaulay durations of 5 years.
      i. Create **two** different portfolios both having Macaulay durations of 5 years.
      ii. Which would you prefer and why?

b. Now you are a bond trader and want to do the following "yield curve" trade: Go long the 30-year bond and go short on the 2-year bond. Suppose you want to buy $10 million face value of the 30-year bond.
   i. How much of the 2-year bond do you have to short to have a modified duration of zero?
   ii. What is the convexity of this portfolio?
   iii. What is this trade betting on?
   iv. Using only modified duration, find the approximate percentage change in value of the portfolio with respect to an immediate drop of 50 basis points in yields (assuming a parallel shift)?
   v. Using only convexity, find the approximate percentage change in value of this portfolio with respect to an immediate drop of 50 basis points in yields (assuming a parallel shift)?

4. Suppose the stock price process follows the dynamics of a geometric Brownian motion under the risk neutral probability measure (Ω, *F*, Q): $dS_t = S_t(rdt + \sigma dW_t)$, where interest rate and volatility parameters r, σ are positive constants, and W_t is a standard Brownian motion. Let $dX_t = \delta_t dS_t + (X_t - \delta_t S_t) rdt$ be a self-financing portfolio that invests δ_t shares in the stock at any time t. Should the wealth of any self-financing portfolio discounted by the stock price $\frac{X_t}{S_t}$ be a martingale? What is the intuition behind your answer? Now

   a. Write down the definition of a martingale applied to the discounted process $\frac{X_t}{S_t}$. (Note that it is very difficult to draw conclusions directly from this definition.)

   b. Use the Ito-Doeblin formula to find out the stochastic differential equation (SDE) for $d\frac{1}{S_t}$ and then $d\frac{X_t}{S_t}$ (product rule). Use your knowledge about this SDE to justify your answer to the above question. (Note the derivatives of certain power functions are $(\frac{1}{x})' = -\frac{1}{x^2}, (\frac{1}{x^2})' = -\frac{2}{x^3}$.)

5. Suppose that S is a random variable which is defined on $[0, \infty)$ and whose probability density function is

$$G(S) = \frac{1}{\sqrt{2\pi} bS} e^{-\left[\ln(S/a) + b^2/2\right]^2 / 2b^2},$$

a and b being positive numbers.

a. Show that

i. $$\int_0^c G(S)\, dS = N\left(\frac{\ln(c/a) + b^2/2}{b} \right);$$

ii. $$\int_0^c SG(S)\, dS = \partial N\left(\frac{\ln(c/a) - b^2/2}{b} \right)$$

where $$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\xi^2/2} d\xi.$$

b. As we know, the price of a European call option is given by

$$c(S,t) = e^{-r(T-t)} \int_0^\infty \max(S' - E, 0)\, G(S', T; S, t)\, dS',$$

where

$$G(S', T; S, t) = \frac{1}{\sigma\sqrt{2\pi(T-t)}\, S'} e^{-\left\{\ln S' - \left[\ln S + (r - D_0 - \sigma^2/2)(T-t)\right]\right\}^2 / 2\sigma^2 (T-t)}$$

Based on the result above, derive the Black-Scholes formula for a European call option.

6. In the Exponential-Vasicek (EV) short-rate model, the natural logarithm of the short-rate follows an Ornstein-Uhlenbeck process, y, under the risk-neutral measure Q. That is, the short-rate, r, is defined by

$$r(t) = \exp(y(t))$$

with

$$dy(t) = \theta - \alpha y(t) \;\; dt + \sigma dW(t), \quad y(0) = y_0,$$

where θ, α and σ are positive constants, and $W(t)$ is a standard Brownian motion under Q.

   a. Use Ito’s lemma to derive the stochastic differential equation governing r. (That is, using Ito’s lemma, what is $dr(t)$?)
   b. In the EV model, is the short-rate lognormally distributed? Does this differ from the standard Vasicek short-rate model? Explain.