Directions: This exam consists of 8 questions. In order to pass the exam, you must answer each question. Where there have been multiple professors teaching the same class, we have provided a question from each (A. and B.). You should select the question most familiar for you, but don’t answer both. **If you do answer both, only A. will be graded.**

Do not write your name on your answer sheet or on any exam page other than this cover sheet. You have been assigned an identification letter below. Write this identification letter on each of your answer pages and on each of the test sheets.

Please note that we will not accept any answer sheet pages with your name on them; if your name is on the sheet we will not grade it.

Your identification letter for this exam is: _______________________

Academic Honesty Statement: All students must comply with University policies on academic integrity. Any student violating these policies, as defined on pages 262-263 of the Graduate Catalog, will be referred to the University administration for disciplinary action. Sanctions for academic dishonesty include, but are not limited to, failure of this exam, suspension, or expulsion from the University. By signing below you certify that the answers you place on this exam are your own and that you have not received help from others.

I hereby certify that the work on this exam is my own. I also certify that I am aware that the test cannot be turned in after 4:00 pm and that I must place my exam identification letter on my exam sheets.

Student name (print) _________________________________

Student Signature: _________________________________
1. There are $S = 3$ states and $N = 3$ securities with payoffs

\[
D = \begin{pmatrix}
20 & 10 & 10 \\
30 & 20 & 10 \\
40 & 30 & 10
\end{pmatrix}
\]

and prices $p = [27, 18, 9]'$. You may find it useful to know that

\[
\begin{pmatrix}
20 & 30 & 40 & 27 \\
10 & 20 & 30 & 18 \\
10 & 10 & 10 & 9
\end{pmatrix}
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 2 & \frac{9}{10} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

(a) Describe the set of all state-price vectors.

(b) Is this market complete? Explain.

(c) Does this market permit arbitrage opportunities? Explain.

(d) Consider two additional payoffs in this market, $X = [150, 200, 250]'$ and $Y = [100, 275, 400]'$. Which of the following statements is/are true for all pairs of distinct state-price vectors $\Psi_1$ and $\Psi_2$. (Distinct means that $\Psi_1 \neq \Psi_2$.)
2. There were two professors who taught FINN/ECON6219. We have provided one question from each professor below. Please answer either question A OR question B below. Do not answer both A AND B.

A. Show the details of your work!

Given a stock index monthly return time series data and you want to check whether the market is efficient or not. You decide to perform a unit-root test. Present your null hypothesis, alternative hypothesis and the decision rule.

B. Show the details of your work! Answer all parts.

The three ARCH/GARCH models below were estimated for one year of daily S&P 500 Index returns. In each model $\varepsilon_t$ is a Gaussian white noise series.

a) Identify the type and order of each of the ARCH/GARCH model below.

b) Based on the given information, determine which model is the best for the given time series and explain why.

Model I

\[
\begin{align*}
    r_t &= -0.000262 + a_t \\
    a_t &= \sigma_t \varepsilon_t \\
    \sigma_t^2 &= 1.301714 - 0.000005a_{t-1}
\end{align*}
\]

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Std.Error</th>
<th>t-value</th>
<th>t-prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cst(M)</td>
<td>-0.000262</td>
<td>0.00075432</td>
<td>-0.3473</td>
</tr>
<tr>
<td>Cst(V)</td>
<td>1.301714</td>
<td>0.13065</td>
<td>9.963</td>
</tr>
<tr>
<td>ARCH(Alpha1)</td>
<td>-0.000005</td>
<td>0.060717</td>
<td>-0.000005</td>
</tr>
<tr>
<td>AIC</td>
<td>-6.084931</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Q-Statistics on Standardized Residuals [P-Values in brackets]

Q(10) = 7.87960  [0.6405963]
Q(15) = 13.6750  [0.5503078]
Q(20) = 14.6565  [0.7957146]

Q-Statistics on Squared Standardized Residuals [P-Values in brackets]

Q(10) = 55.9756  [0.0000000]
Q(15) = 65.7094  [0.0000000]
Q(20) = 81.4076  [0.0000000]
Model II

\[ r_i = -0.000952 + a_i \]
\[ a_i = \sigma_i \varepsilon_i \]
\[ \sigma_i^2 = 0.036885 + 0.104814 a_{i-1}^2 + 0.870776 \sigma_{i-1}^2 \]

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Std.Error</th>
<th>t-value</th>
<th>t-prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cst(M)</td>
<td>-0.000952</td>
<td>-1.581</td>
<td>0.1151</td>
</tr>
<tr>
<td>Cst(V)</td>
<td>0.036885</td>
<td>1.803</td>
<td>0.0725</td>
</tr>
<tr>
<td>ARCH(Alpha1)</td>
<td>0.104814</td>
<td>3.375</td>
<td>0.0009</td>
</tr>
<tr>
<td>GARCH(Beta1)</td>
<td>0.870776</td>
<td>28.47</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

AIC = -6.216553

Q-Statistics on Standardized Residuals [P-Values in brackets]

Q(10) = 7.24759 [0.7018829]
Q(15) = 10.9317 [0.7574193]
Q(20) = 11.5096 [0.9319230]

Q-Statistics on Squared Standardized Residuals [P-Values in brackets]

Q(10) = 19.7115 [0.0114845]
Q(15) = 24.7515 [0.0248805]
Q(20) = 29.5278 [0.0422976]

Model III

\[ r_i = -0.001171 + 2.263867 \sigma_i^2 + a_i \]
\[ a_i = \sigma_i \varepsilon_i \]
\[ \sigma_i^2 = 0.036875 + 0.104756 a_{i-1}^2 + 0.870846 \sigma_{i-1}^2 \]

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Std.Error</th>
<th>t-value</th>
<th>t-prob</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cst(M)</td>
<td>-0.001171</td>
<td>-0.9920</td>
<td>0.3222</td>
</tr>
<tr>
<td>Cst(V)</td>
<td>0.036875</td>
<td>1.815</td>
<td>0.0708</td>
</tr>
<tr>
<td>ARCH(Alpha1)</td>
<td>0.104756</td>
<td>3.385</td>
<td>0.0008</td>
</tr>
<tr>
<td>GARCH(Beta1)</td>
<td>0.870846</td>
<td>28.71</td>
<td>0.0000</td>
</tr>
<tr>
<td>ARCH-in-mean(var)</td>
<td>2.263867</td>
<td>10.549</td>
<td>0.2146</td>
</tr>
</tbody>
</table>

AIC = -6.208769

Q-Statistics on Standardized Residuals [P-Values in brackets]

Q(10) = 7.40316 [0.6869142]
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Q(15) = 11.0453  [0.7493795]
Q(20) = 11.6373  [0.927958]

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Q-Statistics on Squared Standardized Residuals [P-Values in brackets]
Q(10) = 20.0398  [0.0101866]
Q(15) = 24.8263  [0.0243264]
Q(20) = 29.7673  [0.0397707]

3. Answer the following question related to MATH6203 material. Show the details of your work!

**Problem on Stochastic Calculus for Finance.** What are the first and second fundamental theorems of mathematical finance? Give an example of an incomplete market model with no arbitrage: write down stock dynamics \( dS_t = \ldots dt + \ldots dW_t \) and apply the fundamental theorem to explain.

4. There were two professors who taught MATH6202. We have provided one question from each professor below. Please answer either question A OR question B below. Do not answer both A AND B.

   **A. Show the details of your work!**

   **Problem on PDE for Finance.** Derive the Black-Scholes PDE for derivative pricing. What is its relationship to the heat equation? How does this relationship help to find the Black-Scholes formula for the price of a European Call option?
4. B. Show the details of your work! (Disregard the number of points assigned to each part.)

5. The formulation of the American call option as a free-boundary value problem is as follows.

\[
\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D)S \frac{\partial C}{\partial S} - rC = 0, \quad 0 \leq S \leq S_f, \quad 0 \leq t \leq T
\]

\[
C(S, T) = \max(S - E, 0) \quad 0 \leq S \leq S_f(T)
\]

\[
C(S_f(t), t) = S_f(t) - E \quad 0 \leq t \leq T
\]

\[
\frac{\partial C(S_f, 0)}{\partial S} = 1 \quad 0 \leq t \leq T
\]

\[
S(T) = \max(E, \frac{pe^T}{D})
\]

a) Formulate the ODE for the perpetual call option \( C_{\infty}(S) \). (6 pts)

b) Find a solution of the perpetual option in the form \( S^\alpha \). (6 pts)

c) Seek a general solution in the form of \( AS^\alpha + BS^\beta \), and then show that in the context of the problem, one of the coefficients (\( A \) or \( B \)) must be zero. (4 pts)

d) Find the solution \( C_{\infty}(S) \). (5 pts)
5. There were two professors who taught MATH6204. We have provided one question from each professor below. Please answer either question A OR question B below. Do not answer both A AND B.

**A. Show the details of your work!**

Problem:

Explicit finite differences and trinomial trees. The Black-Scholes equation \( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rSV \frac{\partial V}{\partial S} - rV = 0 \) can be transformed into

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial x^2} + (r - \frac{1}{2} \sigma^2) \frac{\partial V}{\partial x} - rV = 0,
\]

where \( x = \log S, \) \( t \in [0, T], \) time \( t = 0 \) is the present time, and time \( T \) is the maturity date. \( V[S_t, t] \) is the time-\( t \) value of an European call and \( V[S_T, T] = \max\{S_T - K, 0\}. \)

1. To compute \( V[S_0, 0], \) show how you can apply the explicit finite difference method (FDM) directly to the log-transformed partial differential equation given above.

2. From your answer to question (1), show that the explicit FDM is actually a direct application of trinomial trees.

**B. Show the details of your work!**

1. Show that if

\[
\max_{0 \leq m \leq M} \frac{x_m^2 (1 - x_m)^2 \sigma_m^2 \Delta \tau}{2} \lesssim \frac{1}{2},
\]

then for the scheme with variable coefficients

\[
\frac{u_{m+1}^n - u_m^n}{\Delta \tau} = \frac{1}{2} [x_m (1 - x_m) \sigma_m] \left[ \frac{\sigma_m^2}{\Delta \tau} \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2} \right. \\
+ (r - D_0)x_m (1 - x_m) \frac{u_{m+1}^n - u_{m-1}^n}{2 \Delta x} \\
\left. - [r (1 - x_m) + D_0 x_m] u_{m}^n \right],
\]

the condition \( |\gamma(x_m, \tau^n)| \leq 1 + O(\Delta \tau) \) is satisfied for any \( x_m = m/M \in [0, 1]. \) (When you prove this result, you should derive the stability condition for explicit schemes by yourself.)
6. If you took FINN6211, answer question A. If you took FINN6058 instead of FINN6211, answer question B. Do not answer both A AND B.

A. Show the details of your work!

1. Answer the following questions:
   a. In what sense is a mortgage similar to a callable bond?
   b. In what sense is a mortgage different from a callable bond?
   c. A recent article in a financial publication reported the following fact about mortgage-backed securities: "When interest rates decline, both IOs (Interest Only tranches) and inverse IOs decline in price, but IOs suffer more severely." Explain why this may be the case.
   d. What are the most important differences between sovereign and corporate bonds?
   e. You are given the following information on three zero coupon bonds. Suppose you have $100,000 to invest.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Spot Rate (%)</th>
<th>Dollar Duration ($)</th>
<th>Dollar Convexity ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5 years</td>
<td>3.9%</td>
<td>0.481</td>
<td>0.472</td>
</tr>
<tr>
<td>3 years</td>
<td>4.9%</td>
<td>2.53</td>
<td>8.65</td>
</tr>
<tr>
<td>5 years</td>
<td>5.6%</td>
<td>3.69</td>
<td>19.7</td>
</tr>
</tbody>
</table>

i. What par amount would you invest in each zero coupon bond if you decide to hold a 3-year bullet portfolio?
ii. What par amount would you invest in each zero coupon bond if you decide to hold a barbell portfolio with equal par amount in the 5-year and 6-month zero?
iii. How does the value of your portfolio in (i) and (ii) change if 6-month zero coupon rates decrease by 10 basis points, 3-year zero coupon rates increase by 5 basis points, whereas 5-year zero coupon rates remain unchanged? Use both duration and convexity to approximate the changes in portfolio value.
iv. Construct a cash and dollar duration neutral butterfly. What par amount will you invest in each wing?
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B. Show the details of your work!

1. The Cox-Ingersoll-Ross (CIR) short rate model assumes that the short rate follows

\[ dr(t) = k(\theta - r(t))\,dt + \sigma \sqrt{r(t)}\,dW(t), \quad r(0) = r_0, \]

where \( r_0, k, \theta, \) and \( \sigma \) are positive constants, and \( W(t) \) is a standard Brownian motion under the risk-neutral measure. In the CIR model, the time \( t \) price for a \( T \)-maturity zero coupon bond is given by

\[ P(t, T) = A(t, T)e^{-B(t,T)r(t)}, \]

where

\[
A(t, T) = \left[ \frac{2h \exp\{(k + h)(T - t)/2\}}{2h + (k + h)\exp\{(T - t)h\} - 1} \right]^{2\theta/\sigma^2} \\
B(t, T) = \frac{2\exp\{(T - t)h\} - 1}{2h + (k + h)\exp\{(T - t)h\} - 1} \\
h = \sqrt{k^2 + 2\sigma^2}.
\]

(a) Derive the fundamental PDE with appropriate terminal condition that \( P(t, T) \) satisfies.

(b) Derive the pair of ODEs that can be solved to find \( A(t, T) \) and \( B(t, T) \).

(c) Is the CIR model an affine term structure model? Explain.

(d) What is the asymptotic mean short rate, \( \lim_{t \to \infty} E_r[r(t)] \)? In other words, to what short rate does \( r(t) \) revert?

(e) What is the major advantage of the CIR short rate model over the Vasicek short rate model?
7. Students could select from Statistical Techniques or Econometrics to satisfy required coursework. We have provided questions from each course on the next pages. Please select to answer either questions related to Statistical Techniques (A.) or Econometrics (B.) Do not answer both A AND B. Note: Econometrics has three pages of material. All parts must be answered if you select to do that material (B.).

A. Statistical Techniques in Finance. Show details of your work! Answer all parts.

2. a) Logarithmic return log X has \( N(\mu, \sigma^2) \) distribution. Derive the skewness and kurtosis of \( X \).

b) Let \( Y_t \) be a stationary AR(2) process

\[
Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \epsilon_t
\]

where \( \epsilon_t \) is a Gaussian white noise sequence. Show that the ACF of \( Y_t \) satisfies the equation

\[
\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2)
\]

for all values of \( k > 0 \).
7. Continued. Students could select from Statistical Techniques or Econometrics to satisfy required coursework. If you answered A on the previous page, do not complete this section too. If you omitted A on the previous page, you must answer all parts of B below.

B. Econometrics. Show details of your work! Answer all parts.

There are competing stories concerning the real estate bubble and whether the bubble was predictable. The following is a time plot of the monthly Case-Shiller House Price Index for the city of Las Vegas, NV from January 1987 through April 2009:

The following three graphs depict the Dickey-Fuller test statistic calculated using a rolling three year (36 month) window.

7. B. Econometrics continued on next page
7. Econometrics Continued

7. B. Econometrics continued on next page
7. Econometrics Continued

The following are the critical values for the Dickey-Fuller test statistic:

<table>
<thead>
<tr>
<th></th>
<th>1% Critical</th>
<th>5% Critical</th>
<th>10% Critical</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-3.682</td>
<td>-2.972</td>
<td>-2.618</td>
</tr>
</tbody>
</table>

(a) If real-estate appreciates at the rate of inflation, what does this imply about the stationarity of the nominal price series? Explain.

(b) If a housing bubble is characterized by housing prices appreciating “too fast” what might this imply about the level of integration of housing prices during the bubble?

(c) Using the graph of the levels of the price index, what can you postulate about the stationarity of the price series over time? Explain.

(d) Given the Dickey-Fuller test statistics presented in the three graphs above, what is the general level of integration of the Las Vegas house price index over the sample period? Explain.

(e) Consider the following time plot of the one-month differences in the Las Vegas housing index and the fitted values from the following regression:

\[ \Delta Price_t = \alpha + \sum_{k=1}^{10} \beta_k TIME^k + \epsilon \]

where \( TIME \) is a time index \( TIME \in [1, 268] \):

Discuss the predictability of the existence of a real estate “bubble” in Las Vegas given the tenth-order polynomial's fitted values.
8. Answer the following question related to FINN6210 material. Show the details of your work! Answer all parts.

Consider a 6-month European call option on a stock with strike price $48. The constant risk-free rate is 5%.

a) Use a 2-step binomial tree to find the price of the call, if the stock has price $50 and volatility 40%.

b) Suppose you sell 100 of these calls. How many shares of stock should you initially purchase to perfectly hedge your position?