University of North Carolina at Charlotte
Mathematical Finance Program
Comprehensive Exam

Fall, 2014

Directions: This exam consists of 8 questions. In order to pass the exam, you must answer each question. Where there have been multiple professors teaching the same class, we have provided a question from each (A. and B.). You should select the question most familiar for you, but don't answer both. <u>If you do answer both, only A. will be graded.</u>

Do not write your name on your answer sheet or on any exam page other than this cover sheet. You have been assigned an identification letter below. Write this identification letter on each of your answer pages *and* on each of the test sheets.

<u>Please note that we will not accept any answer sheet pages with your name on them; if your name is on the sheet we will not grade it.</u>

Your identification letter for this exam is: ______________________

Academic Honesty Statement: All students must comply with University policies on academic integrity. Any student violating these policies, as defined on pages 262-263 of the Graduate Catalog, will be referred to the University administration for disciplinary action. Sanctions for academic dishonesty include, but are not limited to, failure of this exam, suspension, or expulsion from the University. By signing below you certify that the answers you place on this exam are your own and that you have not received help from others.

I hereby certify that the work on this exam is my own. I also certify that I am aware that the test cannot be turned in after 4:00 pm and that I must place my exam identification letter on my exam sheets.

Student name (print) ________________________________

Student Signature: __________________________________

Exam Identification Letter: _____________

Exam Policy for UNC Charlotte Master of Science in Mathematical Finance Program (Fall 2014)

1. Professors/proctors have the right to assign seats for students. If seats are not assigned, students need to spread out in the classroom.
2. During exam, students can use the materials allowed or provided by the professors. No other materials are allowed.
3. Only pen, pencil, eraser and a simple calculator are allowed during exam.
4. Students must use their own calculators. If a student wants to borrow a calculator, he or she can only borrow from the professor/proctor if available.
5. Students cannot use scratch paper or paper with notes/formulas during exam unless the professors/proctors allow them to do so.
6. Students will be given sufficient time to use the restroom before exam starts. ONLY in an urgent situation where the student must go to the restroom, he or she needs to obtain approval from the professor/proctor before leaving the exam room.

The above exam policy is consistent with the Code of Student Academic Integrity of the University of North Carolina at Charlotte. Please make sure to review carefully the entire policy at this link: http://legal.uncc.edu/policies/up-407.

The program has a zero tolerance policy toward any violations of the UNC Charlotte Code of Student Academic Integrity and the exam policies set forth by the professors. Professors/Proctors have the full right to ask the student who violates the academic integrity standards leave the exam room immediately. Any student, who violates the academic integrity standard and is consequently suspended or terminated, will not be readmitted to the program through reinstatement.

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1. **There were two professors who taught FINN/ECON6203. We have provided one question from each professor below. Please answer either question A OR question B below. Do not answer both A AND B.**

A.

1. There are $S = 3$ states and $N = 3$ securities with payoffs

$$D = \begin{pmatrix} 20 & 10 & 10 \\ 30 & 20 & 10 \\ 40 & 30 & 10 \end{pmatrix}$$

and prices $p = [27, 18, 9]'$.

(a) Describe the set of all state-price vectors.

(b) Is this market complete? Explain.

(c) Does this market permit arbitrage opportunities? Explain.

(d) Consider two additional payoffs in this market, $X = [150, 200, 250]'$ and $Y = [100, 275, 400]$. Which of the following statements is/are true for all pairs of distinct state-price vectors Ψ_1 and Ψ_2. (Distinct means that $\Psi_1 \neq \Psi_2$.)

(a) $\Psi_1 \cdot X = \Psi_2 \cdot X$

(b) $\Psi_1 \cdot Y = \Psi_2 \cdot Y$

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B.

Consider a two-period Binomial model in which time t = 0, 1, 2, and each time period represents one year. There are two basic securities in the market. The first one is a risk-free asset with constant interest rate 2% (annually compounding). The second one is a risky asset, and at each time period, this asset's price can either move up 30% or move down 30%. The time zero price of the risky asset is $10. One investment bank XYZ sells a European-type derivative with two years to maturity, and this derivative (structured product) can be exercised only at time t=2. When the derivative is exercised at time t, the owner of the derivative receives, from XYZ,

$$\max\{S(t) - 6, 0\} - \max\{S(t) - 12, 0\}$$

where $S(0), S(t)$ are the risky asset's price at time 0 and time t, respectively,

a) Determine the fair price of this derivative at time t=0. In other words, ignoring transaction costs, how much XYZ should charge to the buyer at time t=0 to break even?

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b) Assume XYZ charges $4 for this derivative. Does XYZ make money for sure without implementing a hedging strategy? If necessary, construct a hedging strategy, or equivalently, a hedging portfolio by these two basic assets, from time t=0 and t=1.

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2. Answer following question related to FINN/ECON6219 material. Show the details of your work! Answer all parts in this question.

Let u_t denote an identically and independently distributed (i.i.d.) random variable that has a mean of 0 and a variance of 1.

(a) Suppose that c_t is generated by the model

$$c_t = \phi_1 c_{t-1} + \phi_2 c_{t-2} + u_t.$$

Show that this model can be expressed as a first-order vector autoregressive process (VAR(1) process).

(b) The process in part (a) can also be expressed as

$$(1 - \lambda_1 L)(1 - \lambda_2 L)c_t = u_t.$$

where L denotes the lag operator. Suppose that $\lambda_1 = 1$ and $|\lambda_2| < 1$. What kind of process does $\Delta c_t = c_t - c_{t-1}$ follow? Be as specific as possible.

(c) Suppose that $\lambda_1 = 1$ and $\lambda_2 = 0.9$. What is the expected value of Δc_{t+2} given that $\Delta c_t = 2$? Your answer should be a number.

(d) Show that the process in part (a) implies that

$$\Delta c_t = \pi c_{t-1} + \gamma \Delta c_{t-1} + u_t,$$

where π and γ are functions of ϕ_1 and ϕ_2. Explain how to test the null hypothesis $\pi = 0$. Be as specific as possible.

(e) Now suppose that $|\lambda_1| < 1$ and $|\lambda_2| < 1$, but λ_1 is an imaginary number, such as $0.5 + 0.4i$. Explain how the partial autocorrelation function (PACF) of the process would behave in this case. Be as specific as possible.

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3. Answer the following question related to FINN 6210 material. Show the details of your work! Answer all parts in this question.

Suppose you are long a call C_1 with strike K_1, long a call C_3 with strike K_3, and short **two** calls C_2 with strike $K_2 = \frac{K_1+K_3}{2}$. The three calls are European, have the same maturity, and have the same underlying stock.

1. Draw the payoff diagram. Be as detailed as possible.

2. Show that if the call prices satisfy $C_2 < \frac{C_1+C_3}{2}$ then there is an arbitrage opportunity. Carefully explain the arbitrage strategy and show that it generates a riskless profit.

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4. **Answer the following question related to FINN 6210 material. Show the details of your work! Answer all parts in this question.**

a) For each of the following statements, provide in your answer: (1) if you AGREE or DISAGREE with the statement, and (2) one to two sentences explanation for why (if you agree) or why not (if you disagree).

a1) The modified duration of a Treasury bill equals its time to maturity.

a2) The coupon yield curve always lies on or above the spot rate curve because the coupon payments enhance the yield to maturity of a coupon bond.

a3) Suppose you have a short position in a 30-year 6% coupon bond and a long position in a zero-coupon bonds with exactly the same market value and duration. If all spot rates fall by 20 basis points, your net position will rise in value.

b) You are given the following information on three bonds:

Maturity	YTM	Bond Price	$ Duration	Quantity
2 years	5.5%	100.6882	195.01	q_S
5 years	6.5%	98.3655	450.14	-1,000
10 years	8%	97.9847	782.18	q_L

We want to construct a butterfly by selling 1,000 contracts of 5-year bonds and buying q_s of 2-year contracts and q_L of 10-year bonds (q_s and q_L represent number of contracts). Use the maturity weighing approach to build your butterfly. In other words, the beta would be (maturity on the middle bond – maturity on the short bond)/(maturity on the long bond – maturity on the middle bond). What would be the q_s and q_L?

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c) As a bond trader, you are very tentative to the changes in interest rate level and often trade on your predictions. Suppose you are confident that the level of interest rates will drop soon and you want to construct a "naïve strategy" to take advantage of this prediction. There are four bonds available for investing (assume $100 par value on all bonds). Suppose yield is expected to drop from the current level of 5% to 4.5%.

Bond	Maturity (years)	Coupon Rate (%)	Yield (%)	Price ($)	Modified Duration (years)	Dollar Duration ($)
1	5	5%	5%	$100.0000	4.38	$437.6032
2	5	6%	5%	104.3760	4.30	448.9094
3	30	5%	5%	100.0000	15.45	1,545.4330
4	30	6%	5%	115.4543	14.91	1,721.4750

If your goal is to maximize the relative capital gain, which bond should you invest and why? Support your answer with calculations of the relative profit and loss for bond 1, 2, 3, and 4 respectively. (Hint: Relative profit and loss = - modified duration × change in yield.)

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5. Answer the following question related to MATH 6203 material. Show the details of your work! Answer all parts in this question.

Problem on Stochastic Calculus for Finance.

Theorem 0.1 (Girsanov Theorem) *Let $(W_t)_{0\le t\le T}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$. Let $(\theta_t)_{0\le t\le T}$ be an adapted process.*

$$\text{Define } Z_t = e^{-\int_0^t \theta_u dW_u - \frac{1}{2}\int_0^t \theta_u^2 du}, \quad \tilde{W}_t = W_t + \int_0^t \theta_u du,$$

and assume that

$$E\int_0^T \theta_u^2 Z_u^2 du < \infty.$$

Then $EZ_T = 1$ and under the probability measure $\tilde{P}$ given by

$$\tilde{P}(A) = \int_A Z_T(\omega) dP(\omega) \text{ for all } A \in \mathcal{F},$$

the process $(\tilde{W}_t)_{0\le t\le T}$ is a Brownian motion.

Question: Let W_t be a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Define the process

$$Z_t = e^{W_t - \frac{1}{2}t}.$$

1. Show that Z_t is a Radon-Nikodým Derivative Process, i.e., it is a positive martingale starting at 1.
2. Use Z_t to define a new probability measure $\tilde{P}$ by $\frac{d\tilde{P}}{dP}|_t = Z_t$, i.e.,

$$\tilde{P}(A) = \int_A Z_T(\omega) dP(\omega) \text{ for all } A \in \mathcal{F}.$$

   Is W_t a Brownian motion under $\tilde{P}$? If the answer is 'yes', show why. If the answer is 'no', write down what is a Brownian motion under $\tilde{P}$.

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6. **Students could select from Cross Section and Time Series Economics (A) or Statistical Techniques in Finance (B) to satisfy required coursework. Please answer either question A OR question B below. Do not answer both A AND B.**

A.

1. Explain the differences between a stationary time series (strong and weak) and a non-stationary time series. What is wrong with using non-stationary data in regression models?

2. You are presented with a model of the following form:

$$\mathit{ln}Y_i = \beta_0 + \beta_1 X1_i + \beta_2 X2_i + \beta_3 \mathit{ln}X3_i + u_i,$$

   where i=1…N, *ln* indicates the natural log and ui is a zero-mean error term. You are asked to assess the model's validity. List and describe how you would go about validating the model.

3. Consider the claim that variable X causes Y and the counterclaim that variable Y causes X. Explain how you can extricate causation in a time-series environment.

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B.

1. a) For the regression model $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$, $i = 1, 2, \ldots$ where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ are i.i.d. $N(0, \sigma_\epsilon^2)$, obtain the least squares estimator of β_0 and β_1.

b) What are the drawbacks of ARMA models? Define heteroscedasticity.

c) For an ARCH(1) model,

$$X_t = \epsilon_t \sqrt{\alpha_0 + \alpha_1 X_{t-1}^2}, \quad t = 1, 2, \ldots, n$$

where $\alpha_0 > 0$ and $0 \leq \alpha_1 < 1$ and ϵ_t is a Gaussian white noise sequence with $\sigma_\epsilon^2 = 1$, derive the conditional least squares estimators of (α_0, α_1) which minimizes $\sum_{t=1}^{n} \{X_t^2 - E[X_t^2 | X_0, \ldots, X_{t-1}]\}^2$.

d) What is the behavior of the process when $\alpha_1 > 1$?

e) State the equations for GARCH (p,q) model.

f) State the equations for a ARIMA(p_A, d, q_A) model with GARCH(p_G, q_G) errors. How many total number of parameter are there in this model?

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7. Answer the following question related to MATH 6202. There are four parts in this question. Answer all parts in this question.

1. (A) State the put-call parity relation for European options. Assume that the underlying pays a continuous dividend D.

(B) Derive the put-call parity stated above.

(C) Show that $c(S,t) \geq Se^{-D(T-t)} - Ee^{-r(T-t)}$.

(D) Deduce that for a non-dividend paying underlying asset (i.e., if $D = 0$), it is never optimal to exercise an American call option at a time $\bar{t}$, with $\bar{t} < T$.

$$c(S,t) = S\, e^{-D(T-t)} N(d_1) - E\, e^{-r(T-t)} N(d_2)$$

$$p(S,t) = E\, e^{-r(T-t)} N(-d_2) - S\, e^{-D(T-t)} N(-d_1)$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S}{E}\right) + \left(r - D_0 + \frac{1}{2}\sigma^2\right)(T-t)\right]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}\,.$$

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8. Answer the following question related to MATH 6204. There are three parts in this question. Answer all parts in this question.

Problem (Implicit FDM and SOR) The Black-Scholes equation $\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$ can be transformed into

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial x^2} - (r - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial x} + rV = 0,$$

where $x = \log S$, $\tau = T - t$, $t \in [0, T]$, time $t = 0$ is the present time, and time T is the maturity date. $V[S_t, t]$ is the time-t value of an European call and $V[S_T, T] = \max\{S_T - K, 0\}$.

1. Apply the implicit finite differences method to discretize the above log-transformed version of the Black-Scholes equation, and show that this exercise can lead to a difference equations system for each time level $\tau \in [1, T]$. Note that such difference equations systems must include information about boundary conditions.
2. Explain how you can apply the SOR (Successive Over-Relaxation) algorithm to solve the difference equations system for each time level.
3. What is the role of the over-relaxation parameter in the SOR algorithm? Explain very briefly.