University of North Carolina at Charlotte  
Mathematical Finance Program  
Comprehensive Exam

Fall, 2013

Directions: This exam consists of 8 questions. In order to pass the exam, you must answer each question. Where there have been multiple professors teaching the same class, we have provided a question from each (A. and B.). You should select the question most familiar for you, but don’t answer both. If you do answer both, only A. will be graded.

Do not write your name on your answer sheet or on any exam page other than this cover sheet. You have been assigned an identification letter below. Write this identification letter on each of your answer pages and on each of the test sheets.

Please note that we will not accept any answer sheet pages with your name on them; if your name is on the sheet we will not grade it.

Your identification letter for this exam is: _______________________

Academic Honesty Statement: All students must comply with University policies on academic integrity. Any student violating these policies, as defined on pages 262-263 of the Graduate Catalog, will be referred to the University administration for disciplinary action. Sanctions for academic dishonesty include, but are not limited to, failure of this exam, suspension, or expulsion from the University. By signing below you certify that the answers you place on this exam are your own and that you have not received help from others.

I hereby certify that the work on this exam is my own. I also certify that I am aware that the test cannot be turned in after 4:00 pm and that I must place my exam identification letter on my exam sheets.

Student name (print) _____________________________

Student Signature: _____________________________
Exam Identification Letter: _____________

Exam Policy for UNC Charlotte Master of Science in Mathematical Finance Program (Summer 2013)

1. Professors/proctors have the right to assign seats for students. If seats are not assigned, students need to spread out in the classroom.
2. During exam, students can use the materials allowed or provided by the professors. No other materials are allowed.
3. Only pen, pencil, eraser and a simple calculator are allowed during exam.
4. Students must use their own calculators. If a student wants to borrow a calculator, he or she can only borrow from the professor/proctor if available.
5. Students cannot use scratch paper or paper with notes/formulas during exam.
6. Students will be given sufficient time to use the restroom before exam starts. ONLY in an urgent situation where the student must go to the restroom, he or she needs to obtain approval from the professor/proctor before leaving the exam room.

The above exam policy is consistent with the Code of Student Academic Integrity of the University of North Carolina at Charlotte. Please make sure to review carefully the entire policy at this link: http://legal.uncc.edu/policies/up-407.

The program has a zero tolerance policy toward any violations of the UNC Charlotte Code of Student Academic Integrity and the exam policies set forth by the professors. Professors/Proctors have the full right to ask the student who violates the academic integrity standards leave the exam room immediately. Any student, who violates the academic integrity standard and is consequently suspended or terminated, will not be readmitted to the program through reinstatement.
1. There were two professors who taught FINN/ECON6203. We have provided one question from each professor below. Please answer either question A OR question B below. Do not answer both A & B.

A. Consider a one-period market with $S = 4$ states, payoff matrix $D$ (securities are in columns), and price vector $p$. Solving $D'\Psi = p$, you find that

$$\Psi = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

is the unique solution. Answer the following questions about this market, carefully explaining your reasoning.

(a) Is this market complete?

(b) Does this market satisfy the Law of One Price?

(c) Does this market admit arbitrage?

(d) Does an equilibrium price measure exist?

(e) What is the risk-free rate in this market?
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B.

State “Put-Call Parity” and present its proof completely by using no-arbitrage argument
2. Answer the following question related to taught FINN/ECON6219 material. Show the details of your work!

Let $u_t$ denote an identically and independently distributed (i.i.d.) random variable that has a mean of 0 and a variance of 1.

(a) Suppose that $z_t$ is generated by the model

$$z_t = \phi_1 z_{t-1} + \phi_2 z_{t-2} + u_t.$$ 

What kind of process is this? Be as specific as possible.

(b) The process in part (a) can also be expressed as

$$(1 - \lambda_1 L)(1 - \lambda_2 L) z_t = u_t.$$

where $L$ denotes the lag operator. Explain in as much detail as you can how to solve for the values of $\lambda_1$ and $\lambda_2$ in terms of $\phi_1$ and $\phi_2$.

(c) Suppose that $\lambda_1 = 1$ and $|\lambda_2| < 1$. What does this tell you about the properties of the process? Given time series data $\{z_t\}_{t=1}^T$, how could you estimate the value of $\lambda_2$? Be as specific as possible.

(d) Now suppose that $\lambda_1 = 0.9$ and $\lambda_2 = 1$. Compute the unconditional mean of $(z_{t+1} - z_t)$, the unconditional variance of $(z_{t+1} - z_t)$, and the first-order autocorrelation coefficient of $(z_{t+1} - z_t)$. 

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(e) Now suppose that $|\lambda_1| < 1$ and $|\lambda_2| < 1$, but $\lambda_1$ is an imaginary number, such as $0.5 + 0.4i$. The autocorrelation function (ACF) of the process is a plot of the autocorrelation coefficient of $z_t$ at lag $j$ versus the value of $j$ (for $j = 1, 2, 3, 4, \ldots$). Explain how the ACF would behave in this case. Be as specific as possible.
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3. Answer the following question related to FINN 6210 material. Show the details of your work! Answer all parts in this question.

Consider a 6-month American put option on a stock when the stock price is $51, the strike price is $50, the volatility is 30%, the risk-free rate is 4%, and the stock is expected to pay a dividend of 2% in 4 months only.

a. Use a 3-step binomial tree to find the price of the option.

b. Use put-call parity for American options along with your answer to part a. to find upper and lower bounds for a 6-month American call option on the stock with strike $50.
4. Answer following question related to FINN 6211 material. Show the details of your work! Answer all parts in this question.

a. A recent article in a financial publication reported the following fact about mortgage-backed securities: "When interest rates decline, both IOs (Interest Only tranches) and inverse IOs decline in price, but IOs suffer more severely." Explain why this may be the case.

b. Explain your understanding of the following terms using a few sentences for each term:
   i. Payment in Kind Bonds
   ii. Rembrandt Bonds
   iii. High yield or Stop-out Yield in the Treasury Auction Process
   iv. Negative Convexity

c. Consider the following three bonds and bond prices. Assume all bonds have a par value of $100 and are semiannual coupon bonds.

<table>
<thead>
<tr>
<th>Coupon</th>
<th>Maturity</th>
<th>Price ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>5/15/2018</td>
<td>96.3750</td>
</tr>
<tr>
<td>7.5%</td>
<td>5/15/2018</td>
<td>103.4043</td>
</tr>
<tr>
<td>15%</td>
<td>5/15/2018</td>
<td>106.0625</td>
</tr>
</tbody>
</table>

Do these prices make sense relative to each other? Demonstrate why or why not.
d. You are given the following information on three zero coupon bonds. Suppose you have $100,000 to invest.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Spot Rate (%)</th>
<th>Dollar Duration ($)</th>
<th>Dollar Convexity ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5 years</td>
<td>3.9%</td>
<td>0.481</td>
<td>0.472</td>
</tr>
<tr>
<td>3 years</td>
<td>4.9%</td>
<td>2.53</td>
<td>8.65</td>
</tr>
<tr>
<td>5 years</td>
<td>5.6%</td>
<td>3.69</td>
<td>19.7</td>
</tr>
</tbody>
</table>

i. What par amount would you invest in each zero coupon bond if you decide to hold a 3-year bullet portfolio?

ii. What par amount would you invest in each zero coupon bond if you decide to hold a barbell portfolio with equal par amount in the 5-year and 6-month zero?

iii. How does the value of your portfolio in (i) and (ii) change if 6-month zero coupon rates decrease by 10 basis points, 3-year zero coupon rates increase by 5 basis points, whereas 5-year zero coupon rates remain unchanged? Use both duration and convexity to approximate the changes in portfolio value.

iv. Construct a cash and dollar duration neutral butterfly. What par amount will you invest in each wing?
5. Answer following question related to MATH 6203 material. Show the details of your work! Answer all parts in this question.

Problem on Stochastic Calculus for Finance. Let \( W_t \) be a standard Brownian motion on a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\). Define the Ito integrals

\[
I_1(t) = \int_0^t s \, dW_s, \quad I_2(t) = \int_0^t W_s \, dW_s
\]

Use the properties for Ito integrals to answer the following questions:

1. Are they martingales?

2. Calculate the quadratic variations and covariation \([I_1, I_1]_t, [I_2, I_2]_t, [I_1, I_2]_t\).

3. Calculate the second moments \(E[(I_1(t))^2], E[(I_2(t))^2]\) using Itô-isometry.

4. Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a deterministic function.
   Use Itô-Doebelin formula to find \(d(I_1(t))^2, d(I_2(t))^2, df(t, (I_1(t))^2, (I_2(t))^2)\).
6. Students could select from Statistical Techniques in Finance (MATH 6201) or Cross Section and Time Series Economics (STAT 6113/ECON 6113) to satisfy required coursework. Please answer the following question. Please answer either question A OR question B below. Do not answer both A AND B.

A.

1. Let \( \{\varepsilon_t\} \) be Gaussian White Noise \((0,1)\) sequence.

   a) For the AR(1) model \( Y_t = \phi Y_{t-1} + \varepsilon_t, \ t = 1, 2, \ldots, n \) where the initial random variable \( Y_0 \) is fixed and \( 0 < \phi < 1 \), derive the least squares estimator of \( \phi \) and the ACF of \( Y_t \).

   b) For the AR(1) model with ARCH(1) errors

   \[
   u_t = 3 + 0.7 u_{t-1} + a_t,
   \]

   \[
   a_t = \varepsilon_t \sqrt{1 + 0.5 a_{t-1}^2}, \ t = 1, 2, \ldots, n,
   \]

   find the mean, variance and ACF of \( u_t \).

   c) State the GARCH (1,1) model.
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B.

ECON6113
Answer the following questions:

1. There are competing stories concerning the real estate bubble and whether the bubble was predictable. The following is a time plot of the monthly Case-Shiller House Price Index for the city of Las Vegas, NV from January 1987 through April 2009:

(a) If real-estate appreciates at the rate of inflation, what does this imply about the stationarity of the nominal price series? Explain.
(b) If a housing bubble is characterized by housing prices appreciating “too fast” what might this imply about the level of integration of housing prices during the bubble?
(c) Using the graph of the levels of the price index, what can you postulate about the stationarity of the price series over time? Explain.
(d) The following three graphs depict the Dickey-Fuller test statistic calculated using a rolling 36 month window.
The following are the critical values for the Dickey-Fuller test statistic:

1% Critical  5% Critical  10% Critical
-3.682       -2.972       -2.618

Given the Dickey-Fuller test statistics presented in the three graphs above, what is the general level of integration of the Las Vegas house price index over the sample period? Explain.

e) Consider the following time plot of the one-month differences in the Las Vegas housing index and the fitted values from the following regression:

\[ \Delta Price_t = \alpha + \sum_{k=1}^{10} \beta_k TIME^k + \epsilon \]

where \( TIME \) is a time index \( TIME \in [1, 268] \):
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7. Answer the following question related to MATH 6202. **Answer all parts in this question.**

a) Describe a chooser option.

b) Suppose that the time when the choice is made is $T_1$. What is the value of the chooser option at this time?

c) Suppose that the options underlying the chooser option are both European with the same strike price $E$. Let $S_1$ be the asset price at time $T_1$, let $T_2$ be the maturity of the options, let $r$ be the risk-free interest rate, and let $D$ be the dividend rate. Derive a formula for the value of the chooser option in terms of a call maturing at $T_1$ and a put with maturing at $T_2$. 
8. Answer the following question related to MATH 6204. Answer all parts in this question.

**Problem**  It is well known that the classical Black-Scholes equation for an European call options

$$
\frac{\partial V(S,t)}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + (r - \delta) S \frac{\partial V(S,t)}{\partial S} - r V(S,t) = 0, \quad 0 < S < \infty, \quad 0 < t < T
$$

can be transformed into a simple heat equation as given below:

$$
\frac{\partial y(x,\tau)}{\partial \tau} = \frac{\partial^2 y(x,\tau)}{\partial x^2}, \quad 0 < \tau < \frac{\sigma^2}{2T}, \quad -\infty < x < \infty
$$

where \( x = \ln(S/K) \) and \( \tau = \frac{\sigma^2}{2} (T - t) \). You are required to do the following:

1. Use the explicit finite difference method to discretize the heat equation into a linear difference system for each time level \( \tau \in (0, \frac{\sigma^2}{2T}) \). Your derived system must contain the boundary conditions at each end of the space domain, and your final results must be presented in matrix form.

2. Use the implicit finite difference method to discretize the heat equation into a linear difference system for each time level \( \tau \in (0, \frac{\sigma^2}{2T}) \). Your derived system must contain the boundary conditions at each end of the space domain, and your final results must be presented in matrix form.

3. How do you compare the explicit to implicit finite difference methods based on their stability problem and computational effort? Present your comparisons briefly, without using any mathematical derivations or proofs.
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