Directions: This exam consists of 8 questions. In order to pass the exam, you must answer each question. Where there have been multiple professors teaching the same class, we have provided a question from each (A. and B.). You should select the question most familiar for you, but don’t answer both. If you do answer both, only A. will be graded.

Do not write your name on your answer sheet or on any exam page other than this cover sheet. You have been assigned an identification letter below. Write this identification letter on each of your answer pages and on each of the test sheets.

Please note that we will not accept any answer sheet pages with your name on them; if your name is on the sheet we will not grade it.

Your identification letter for this exam is: _______________________

Academic Honesty Statement: All students must comply with University policies on academic integrity. Any student violating these policies, as defined on pages 262-263 of the Graduate Catalog, will be referred to the University administration for disciplinary action. Sanctions for academic dishonesty include, but are not limited to, failure of this exam, suspension, or expulsion from the University. By signing below you certify that the answers you place on this exam are your own and that you have not received help from others.

I hereby certify that the work on this exam is my own. I also certify that I am aware that the test cannot be turned in after 4:00 pm and that I must place my exam identification letter on my exam sheets.

Student name (print) ________________________________

Student Signature: ________________________________
Exam Identification Letter: _____________

Exam Policy for UNC Charlotte Master of Science in Mathematical Finance Program (Spring 2013)

1. Professors/proctors have the right to assign seats for students. If seats are not assigned, students need to spread out in the classroom.
2. During exam, students can use the materials allowed or provided by the professors. No other materials are allowed.
3. Only pen, pencil, eraser and a simple calculator are allowed during exam.
4. Students must use their own calculators. If a student wants to borrow a calculator, he or she can only borrow from the professor/proctor if available.
5. Students cannot use scratch paper or paper with notes/formulas during exam.
6. Students will be given sufficient time to use the restroom before exam starts. ONLY in an urgent situation where the student must go to the restroom, he or she needs to obtain approval from the professor/proctor before leaving the exam room.

The above exam policy is consistent with the Code of Student Academic Integrity of the University of North Carolina at Charlotte. Please make sure to review carefully the entire policy at this link: http://legal.uncc.edu/policies/up-407.

The program has a zero tolerance policy toward any violations of the UNC Charlotte Code of Student Academic Integrity and the exam policies set forth by the professors. Professors/Proctors have the full right to ask the student who violates the academic integrity standards leave the exam room immediately. Any student, who violates the academic integrity standard and is consequently suspended or terminated, will not be readmitted to the program through reinstatement.
1. There were two professors who taught FINN/ECON6203. We have provided one question from each professor below. Please answer either question A OR question B below. Do not answer both A AND B.

A. Consider a one-period market with $S = 4$ states, payoff matrix $D$ (securities are in columns), and price vector $p$. Solving $D\Psi = p$, you find that

$$\Psi = \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \\ -1 \end{bmatrix}$$

is the unique solution. Answer the following questions about this market, carefully explaining your reasoning.

(a) Is this market complete?

(b) Does this market satisfy the Law of One Price?

(c) Does this market admit arbitrage?

(d) Does an equilibrium price measure exist?

(e) What is the risk-free rate in this market?
B. Suppose an agent A enters into a one-year forward contract with agent B on a non-dividend-paying stock when the stock price is $100 and the effective risk-free interest is 10% per annum.

(a) What is the forward price?

(b) Suppose one year later, the stock is $80 and agent A doesn’t want to fill her obligation of this forward contract. Agent A ask agent B and agent B also agrees to roll the contract forward to one more year to settle. Assume the interest rate is fixed. What is the delivery price $K$ of this new forward contract?
2. There were two professors who taught FINN/ECON6219. We have provided one question from each professor below. Please answer either question A OR question B below. Do not answer both A AND B.

A.

It is well known that there exists volatility clusters in financial time series, denoted by \( r_t \) which can be written as \( r_t = \mu_t + \sigma_t u_t \) with some mean process \( \mu_t \), \( \sigma_t = \sigma_t u_t \) is the shock, \( \sigma_t^2 = \text{Var}(r_t | I_{t-1}) \) is the conditional variance of \( r_t \) given the information \( I_{t-1} \) up to time \( t - 1 \), and \( u_t \) is a sequence of independently identically distributed (iid) random variables with mean zero and variance \( \sigma_u^2 = 1 \). To characterize the features of volatility, it is common to use the so-called conditional heteroscedastic (ARCH) or generalized conditional heteroscedastic (GARCH) model as follows

\[
\sigma_t^2 = \alpha_0 + \sum_{k=1}^{p} \alpha_k \sigma_{t-k}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \tag{1}
\]

where \( \alpha_0 > 0, \alpha_k \geq 0 \) and \( \beta_j \geq 0 \) and model (1) is commonly called as a GARCH\((p,q)\) model. Of course, a GARCH\((p,q)\) model has its strengths and weaknesses.

(a) Please list all possible weaknesses of a GARCH\((p,q)\) model as you know.

(b) Please list all possible strengths of a GARCH\((p,q)\) model as you know.

(c) Please interpret a GARCH\((p,q)\) model in a layman language. In other words, explain what a GARCH\((p,q)\) model means to people who do not know financial econometrics.

(d) Show that the tail distribution of a GARCH\((1,1)\) process is heavier than that of a normal distribution. To answer this question, you might need to derive the kurtosis of a GARCH\((p,q)\) process.
Assume that the sequence \( (e_t)_{t=1}^{\infty} \) is generated by a stochastic process of the form

\[
e_t = h_t^{1/2} z_t,
\]

where \( z_t \) is i.i.d. \( N(0,1) \) and \( h_t \) is given by

\[
h_t = \omega + \beta h_{t-1} + \alpha e_{t-1}^2.
\]

(a) What kind of process is this? Be as specific as possible.

(b) State the condition that implies that \( e_t \) is covariance stationary. Derive the autocorrelation function (ACF) of \( e_t \).

(c) Suppose \( \omega = 0.1, \beta = 0.7 \) and \( \alpha = 0.2 \). The initial conditions for the process are \( h_0 = 2 \) and \( e_0 = 0 \). Compute the value of \( E(e_2^2|e_1, e_0, h_0) \) for the case in which \( e_1 = 1 \).

(d) Show that \( e_t^2 \) follows an ARMA(1,1) process.

(e) Suppose that \( e_t \) is the error term in a time series regression of the form

\[
y_t = a + bx_t + e_t.
\]

Your friend argues that you can use ordinary least squares (OLS) to estimate \( a \) and \( b \), and that the usual OLS standard errors can be used to conduct valid hypothesis tests, e.g., to test the hypothesis \( b = 0 \). Do you agree or disagree? Explain your answer.
3. Answer the following question related to FINN 6210 material. Show the details of your work! Answer all parts in this question.

You sell a put option with strike price $40 and a put option with strike price $50, and you buy two put options with strike price $45. All options are on the same underlying stock, have the same maturity, and are European-style.

   a. Draw the payoff diagram. What is your maximum liability, and where does it occur?

   b. Show this trading strategy must provide a positive cash flow when first entered (i.e., the cost to implement this strategy is negative—it puts money in your pocket).

   c. Now assume the risk-free rate is 5%, the current stock price is $45 and its volatility is 20%, and the time-to-maturity is 6 months. Use a 2-step binomial tree to estimate the cost.

   d. Is your estimate in part c. likely to be greater than or less than the Black-Scholes price? Explain.
4. Answer following question related to FINN 6211 material. Show the details of your work! Answer all parts in this question.

a) Explain your understanding of the following terms [a few sentences per term is sufficient]:
   a1) Sequential-Pay CMOs
   a2) Reconstitution of STRIPs
   a3) Irredeemable Gilts
   a4) Reopening in the Treasury Market

b) The United States Treasury suspended the auction of its 30-year benchmark [known as the long bond] in October 2001. Explain the possible rationale for this action. Briefly assess the impact of this decision from the perspective of (b1) Insurance companies, (b2) Agencies of government and (b3) Security dealers in the fixed-income markets.

c) Consider the following barbell and bullet portfolios constructed from 10-year, 20-year, and 30-year zeroes:

<table>
<thead>
<tr>
<th></th>
<th>Par</th>
<th>Maturity</th>
<th>Yield</th>
<th>Price</th>
<th>Dollar Duration</th>
<th>Duration</th>
<th>Dollar Convexity</th>
<th>Convexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Barbell</td>
<td>25174.12</td>
<td>10</td>
<td>6.00%</td>
<td>13938.30</td>
<td>135323.3</td>
<td>9.70874</td>
<td>1379510</td>
<td>98.9726</td>
</tr>
<tr>
<td></td>
<td>91897.89</td>
<td>30</td>
<td>6.40%</td>
<td>13884.29</td>
<td>403613.1</td>
<td>29.06977</td>
<td>11928488</td>
<td>859.1356</td>
</tr>
<tr>
<td>Bullet</td>
<td>100000</td>
<td>20</td>
<td>6.50%</td>
<td>27822.59</td>
<td>538936.4</td>
<td>19.37046</td>
<td>10700432</td>
<td>384.5951</td>
</tr>
<tr>
<td>Total/Average</td>
<td></td>
<td></td>
<td>6.30%</td>
<td>27822.59</td>
<td>538936.4</td>
<td>19.37046</td>
<td>13307997</td>
<td>478.3162</td>
</tr>
</tbody>
</table>
c1) Compute the values of the barbell and bullet portfolios after one year assuming the rates on all zeroes stay exactly the same.

c2) Compute the annualized semi-annually compounded rates of return on each of the 3 zeroes (10-, 20-, and 30-year) and on the bullet and barbell portfolios, assuming the rates on all zeroes stay exactly the same.
5. There were two professors who taught MATH 6203. We have provided one question from each professor below. Please answer either question A OR question B below. Do not answer both A AND B.

A.

Problem on Stochastic Calculus for Finance. Let $W_t$ be a standard Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Let $s, t$ be real numbers that satisfy $0 \leq s \leq t$.

1. Calculate the conditional expectation $E[W_t^2 | \mathcal{F}_s]$.

2. Is $W_t^2$ a martingale, supermartingale, submartingale, or none of the above?

3. Find the differential $dW_t^2$ using Itô-Doeblin formula.

4. Calculate the stochastic integral $\int_0^t W_s dW_s$. 
{W_t, t \geq 0} is a standard Brownian motion on a probability space \((\Omega, \mathcal{F}, P)\) with filtration \(\{\mathcal{F}_t, t \geq 0\}\) in the following.

1. Toss a fair coin repeatedly and denote the successive outcomes of the tosses by \(\omega = \omega_1\omega_2\omega_3\ldots\). The symmetric random walk is defined as \(M_k = \sum_{j=1}^{k} X_j\), \(k = 1, 2, \ldots\), \(M_0 = 0\), \(X_j = 1\) when \(\omega_j = H\) and \(X_j = -1\) when \(\omega_j = T\), \(P(H) = P(T) = \frac{1}{2}\). Fix \(t \geq 0\). Show that as \(n \to \infty\) the distribution of the scaled symmetric random walk \(W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt}\) evaluated at time \(t\) converges to the distribution of \(W_t\). Is this a step towards obtaining weak convergence of Binomial model to geometric Brownian motion?
6. Students could select from Statistical Techniques in Finance or Cross Section and Time Series Economics to satisfy required coursework. Please answer either question A OR question B below. Do not answer both A AND B.

A. The Problem for Comprehensive Exam for ECON6113
Suppose that you consider the following autoregressive (AR($p$)) model:

\[ y_t = \sum_{j=1}^{p} \phi_j y_{t-j} + u_t, \quad 1 \leq t \leq T = 100, \quad (1) \]

where $y_t$ is a (weak) covariance-stationary process and $u_t$ is a normal white noise process with mean zero and variance $\sigma^2$. First, you run the ordinary least squares (OLS) estimation for the above AR($p$) model with $p = 9$ and find that $\hat{\sigma}^2 = 2.2$. Also, based on a model selection criterion, say the Akaike Information Criterion (AIC), you might consider the above AR($p$) model with $p = 4$ as an alternative model specification rather than the AR(9) model. The OLS estimation of the AR(4) model gives the variance estimate $\hat{\sigma}^2 = 3.1$. Now, the question is which model would be appropriate, AR(9) or AR(4). Therefore, you need to conduct a statistic test on the AR(9) model versus the AR(4) model by using the F-test.

(a) Please explain why model (1) does not include an intercept.

(b) Please briefly describe what the AIC model selection criterion is. Also, Please briefly describe what the F-test is.

(c) Please use the AIC model selection method to decide whether you should choose the AR(9) model or the AR(4) model based on the information provided.

(d) Please use the F-test to test the AR(9) model versus the AR(4) model based on the given information at the significance level 5%.
B

Math 6201-Statistical Techniques in Finance

1. a) For the regression model $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i, \ i \geq 1$ where $(\epsilon_i)$ are i.i.d. with mean 0 and variance $\sigma^2_\epsilon$, obtain the least squares estimator of $\beta_1$ and its variance.

b) $X$ takes 30 equally spaced values between 1 and 10. Calculate the correlation between $X$ and $X^2$ and Variance Inflation Factors (VIFs) of $X$ and $X^2$. How do you reduce collinearity in a dataset? Calculate the correlation between the new variables and and Variance Inflation Factors(VIFs) of the new variables after reducing collinearity.

c) State the Security Characteristic Line of CAPM.
7. There were two professors who taught MATH 6202. We have provided one question from each professor below. Please answer either question A OR question B below. Do not answer both A AND B.

A.

Show the details of your work !!

1. As we know, \( f = Se^{D_0(T-t)} - Ke^{-r(T-t)} \) is the value of a forward/futures contract. For \( S \) we assume \( dS = \mu S dt + \sigma S dX \), so for \( df \) we have

\[
\begin{align*}
df &= \left[ (\mu + D_0) \left( f + Ke^{-r(T-t)} \right) + rKe^{-r(T-t)} \right] dt \\
+ \sigma \left[ f + Ke^{-r(T-t)} \right] dX
\end{align*}
\]

according to Itô’s lemma.

(a) Consider an option on a forward/futures and let the price of such an option be \( V_1(f, t) \). Derive the PDE for \( V_1 \) by using Itô’s lemma. (Hint: Set \( \Pi = V_1(f, t) - \Delta f \).)

(b) Let \( F = e^{(r-D_0)(T-t)}S \), then for \( f \) we have another expression: \( f = e^{-r(T-t)} \left( Se^{(r-D_0)(T-t)} - K \right) = e^{-r(T-t)} (F - K) \). Define \( V(F, t) = V_1(f(F, t), t) = V_1(e^{-r(T-t)} (F - K), t) \). Derive the PDE for \( V(F, t) \) from the PDE obtained in part a).
B.

a) Describe a chooser option.

b) Suppose that the time when the choice is made is $T_1$. What is the value of the chooser option at this time?

c) Suppose that the options underlying the chooser option are both European with the same strike price $E$. Let $S_1$ be the asset price at time $T_1$, let $T_2$ be the maturity of the options, let $r$ be the risk-free interest rate, and let $D$ be the dividend rate. Derive a formula for the value of the chooser option in terms of a call maturing at $T_1$ and a put with maturing at $T_2$. 
8. Answer the following question related to MATH 6204. There are two parts in this question. Answer all parts in this question.

Problem 1  It is well known that the classical Black-Scholes equation for European style options

\[
\frac{\partial V(S,t)}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + (r - \delta)S \frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0, \ 0 < S < \infty, \ 0 < t < T
\]

can be transformed into a simple heat equation as given below:

\[
\frac{\partial y(x,\tau)}{\partial t} = \frac{\partial^2 y(x,\tau)}{\partial x^2}, \ 0 < \tau < \frac{\sigma^2}{2}T, \ -\infty < x < \infty
\]

where \( x = \ln(S/K) \) and \( \tau = \frac{\sigma^2}{2}(T - t) \). You are asked to answer the following questions:

1. Use the implicit finite difference method (FDM) to discretize the heat equation into a linear system of multiple difference equations.

2. Use the eigenvalue-based stability analysis (known as the “matrix approach”) to demonstrate that the implicit FDM is \textit{unconditionally} stable when one seeks to solve the resulting linear system. Hint: for the stability analysis, you should recall that for a \( K \times K \) tridiagonal matrix \( G \),

\[
G = \begin{bmatrix}
\alpha & \beta & 0 \\
\gamma & \ddots & \ddots \\
& \ddots & \ddots & \beta \\
0 & \gamma & \alpha
\end{bmatrix}_{K \times K}
\]

its eigenvalues, denoted by \( \Lambda_G^k \), are given by
\[ \Lambda_k^G = \alpha + 2\beta \sqrt{\frac{\gamma}{\beta}} \cos \left( \frac{k\pi}{K+1} \right), \quad k = 1, \ldots, K, \]

and that \( \cos 2\theta = 1 - 2\sin^2 \theta \).